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The geometrical framework for Yang–Mills theories

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Abstract
We construct a new geometrical framework for Yang–Mills Lagrangian field theories as an appropriate quotient space of the standard first jet-bundle and investigate the geometrical properties of the resulting mathematical setting. We deduce the field equations from a variational problem formulated through a regular Poincaré–Cartan form, thus ensuring the kinematical admissibility of critical sections. We state a generalized Nöther theorem and explicitly consider the case of the free Yang–Mills field.

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1. Introduction

The main idea of this work arises from the observation that, in many fields of interest, it is possible to describe the dynamical variables of a Lagrangian theory in terms of forms.

As for Yang–Mills fields, for example, one can interpret the dynamical field A as a connection of a suitable principal fibre bundle and the strength field F as its curvature. In this formalism, symmetry and invariance properties play an important role which has been widely investigated in the literature also from a geometrical viewpoint [1–6].

The possibility of using forms to describe fields is, however, not only limited to Yang–Mills fields. Also gravitational fields and classical continuum mechanics, for example, may be described in such a way [7, 8].

In this paper, we construct a new geometrical framework, namely a new bundle, by changing the standard definition of jet-equivalence. The resulting formalism is more suitable to describe Lagrangian field theories depending on the derivatives of the field, only by means of the antisymmetric part of the gradient.

In such a way, the 'inessential' coordinates are cut away from the geometrical construction, since its origin, thus achieving two main goals: firstly, the gauge transformations act in a very simple way on the 'derivatives' of the dynamical field A (namely in a pure tensorial way) and, secondly, the standard Yang–Mills Lagrangian becomes regular. Explicitly, we observe that the Lagrangian

$$L = -\frac{1}{4} F^{\mu}_{ik} F^{ik}_{\mu}$$

is regular, in the sense that there are no longer vertical vector fields belonging to the kernel of its second derivative matrix.

The structure of the paper is as follows. In section 2, we construct the new 'jet-bundle' $\mathcal{J}(E)$ and we study its geometrical properties in detail. More precisely, we define suitable contact 2-forms characterizing the sections which are $\mathcal{J}$-extensions; we introduce the notion of $\mathcal{J}$-prolongability for bundle automorphisms and vector fields and study its relationships with contact forms, trying to extend some classical result of the standard jet-bundle theory to this formalism.

In section 3, we construct the Poincaré–Cartan form $\tilde{\Theta}_L$ in $\mathcal{J}(E)$ and we deduce the field equations from a variational principle, built through $\tilde{\Theta}_L$ on the new space. As we shall see, this variational principle directly furnishes the kinematical admissibility of critical sections as a consequence of the acquired regularity properties of the Yang–Mills Lagrangian. In this connection, we shall introduce a particular choice of local coordinates which are adapted to the structure of the connection and the curvature fields. Doing so, we are able to perform calculations in a very simple form.

In section 4, we shall briefly investigate the relationships between symmetries, Nöther theorem and conserved currents [10–12] in the newer scheme.

Finally, in section 5, we propose an explicit example by considering the free Yang–Mills Lagrangian in Minkowski spacetime. The previously developed geometrical framework is applied, gauge and spacetime symmetries are considered and the standard stress–energy tensor is obtained.

2. The geometrical framework

Yang–Mills theories may be geometrically described by using a principal fibre bundle $P \to M$, with structural group G and base manifold M generally taken to be the spacetime.

The principal connections of $P \to M$ represent the dynamical fields and their pullbacks through local sections of $P \to M$ represent the Yang–Mills fields.

Locally, the physical fields may be regarded as 1-form on open sets $U \subset M$ with values in $\mathcal{G}$, the Lie algebra of G and, therefore, they are local sections of $E := T^{\star}M \otimes \mathcal{G} \to M$.

Referring E to local coordinates $x^i, a^{\mu}_i, i = 1, \ldots, m = \dim M, \mu = 1, \ldots, r = \dim G$, physical fields are then locally described as

$$x \to a^{\mu}_i(x)\, \mathrm{d}x^i|_x \otimes \underline{\mathrm{e}}_{\mu} \tag{2.1}$$

$\underline{\mathrm{e}}_{\mu}$ being a given basis of $\mathcal{G}$.

In the present paper we focus our attention on a local analysis of the theory, supposing a (local) trivialization of P is fixed once and for all. We remark that this choice is not a real restriction since the changes of the local trivialization give rise to *unphysical* changes of the Yang–Mills fields represented by gauge transformations, in turn recovered as (local) active automorphisms of E.

A global approach to the formalism proposed here is also possible and its detailed development is the main argument of a forthcoming paper [9], where all the present geometrical constructions will be built on the usual connection bundle $j_1(P, M)/G \to M$.

According to these arguments, we shall work directly on the space E. To start with, we denote by $j_1(E)$ the first jet-bundle associated with the fibration $E \to M$ and refer it to local jet-coordinates x^i, a^μ_i, a^μ_{ij}. Up to a change of basis in $\mathcal{G}$, the above jet-coordinates undergo the transformation laws

$$\bar{x}^i = \bar{x}^i(x^j) \qquad \bar{a}^\mu_i = a^\mu_j \frac{\partial x^j}{\partial \bar{x}^i} \qquad \bar{a}^\mu_{ij} = a^\mu_{ks} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} + a^\mu_h \frac{\partial^2 x^h}{\partial \bar{x}^i \partial \bar{x}^j}. \tag{2.2}$$

Let us define in $j_1(E)$ the following equivalence relation: given $z_1 = \left(x^i, a^\mu_i, a^\mu_{ij}\right)$, $z_2 = \left(x^i, a^\mu_i, \hat{a}^\mu_{ij}\right) \in j_1(E)$, both fibred on the same point $e = \left(x^i, a^\mu_i\right) \in E$, we say that $z_1 \sim z_2 \Leftrightarrow \left(a^\mu_{ij} - a^\mu_{ji}\right) = \left(\hat{a}^\mu_{ij} - \hat{a}^\mu_{ji}\right)$. Transformation laws (2.2) ensure that the above equivalence relation is independent of the choice of coordinates, being

$$\left(\bar{a}^\mu_{ij} - \bar{a}^\mu_{ji}\right) = \left(a^\mu_{ks} - a^\mu_{sk}\right) \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j}. \tag{2.3}$$

Geometrically speaking, the introduced equivalence relation means that if σ_1 and σ_2 are two sections representing z_1 and z_2 respectively, then $z_1 \sim z_2 \Leftrightarrow \mathrm{d}\sigma_1|_{\pi(z_1)} = \mathrm{d}\sigma_2|_{\pi(z_2)}$, $\pi : j_1(E) \to M$ indicating the natural projection.

Then, we define the quotient space $\mathcal{J}(E) := j_1(E)/\sim$. The main difference of the vector bundle $\mathcal{J}(E)$ with respect to $j_1(E)$ is the fact that the first-order contact condition between sections is calculated using the exterior differential, since sections are 1-forms.

$\mathcal{J}(E)$ may be endowed with a set of local coordinates $x^i, a^\mu_i, \tilde{A}^\mu_{ij} := \frac{1}{2}\left(a^\mu_{ij} - a^\mu_{ji}\right)$ $(i < j)$.

Transformation laws (2.2), (2.3) imply the identification

$$\mathcal{J}(E) \simeq (T^*M \otimes \mathcal{G}) \times_M (\Lambda^2(T^*M) \otimes \mathcal{G}) \tag{2.4}$$

as well as the natural immersion

$$i : \mathcal{J}(E) \to (T^*M \otimes \mathcal{G}) \times_M ((T^*M \otimes T^*M) \otimes \mathcal{G}). \tag{2.5}$$

Referring $(T^*M \otimes \mathcal{G}) \times_M ((T^*M \otimes T^*M) \otimes \mathcal{G})$ to local coordinates x^i, a^μ_i, A^μ_{ij} $(\forall i, j = 1, \ldots, m)$ the submanifold $\mathcal{J}(E)$ is locally described as $A^\mu_{ij} = -A^\mu_{ji}$.

Alternatively, the manifold $\mathcal{J}(E)$ may be constructed using the distribution $\mathcal{D}$ formed by the totality of vertical vectors $V^\sigma_{pq} \frac{\partial}{\partial a^\sigma_{pq}}$ on $j_1(E)$, symmetric in the indices p and q, i.e. satisfying the condition $V^\sigma_{pq} = V^\sigma_{qp}$. In fact, $\mathcal{D}$ is involutive and the leaf space $j_1(E)/\mathcal{D}$ of the foliation generated by $\mathcal{D}$ identifies with the manifold $\mathcal{J}(E)$ introduced above.

As a further remark, we note that the space $\mathcal{J}(E)$ may be put in relation with the semi-holonomic bundle [13] associated with the principal fibre bundle $P \to M$.[4] This fact allows us also to build a global approach which is not one of the aims of the current paper.

Now, we shall see that some fundamental geometrical structures and constructions of the first jet-bundle $j_1(E)$ may be naturally extended to the newly defined manifold $\mathcal{J}(E)$.

2.1. $\mathcal{J}$-extension of sections

Given a section $\sigma : M \to E$, locally expressed as $x \to \left(x^i, a^\mu_i(x)\right)$, we define its first $\mathcal{J}$-extension as $\mathcal{J}\sigma := \rho \circ j_1\sigma$, according to the commutative diagram

$$\begin{array}{ccc} j_1(E) & \xrightarrow{\rho} & \mathcal{J}(E) \\ {\scriptstyle j_1\sigma}\uparrow & & \uparrow{\scriptstyle \mathcal{J}\sigma} \\ M & = & M \end{array} \tag{2.6}$$

ρ indicating the canonical quotient projection and $j_1\sigma$ denoting the standard first jet-extension of σ.

[4] We wish to thank Professor M Modugno for a useful discussion on this point.

Any section $\gamma : M \to \mathcal{J}(E)$ will be called holonomic if and only if there exists a section $\sigma : M \to E$ such that $\gamma = \mathcal{J}\sigma$. In local coordinates, a section γ of the form $x \to \big(x^i, a_i^\mu(x), \tilde{A}_{ij}^\mu(x)\big)$ is holonomic if and only if $\tilde{A}_{ij}^\mu(x) = \frac{1}{2}\big(\frac{\partial a_i^\mu(x)}{\partial x^j} - \frac{\partial a_j^\mu(x)}{\partial x^i}\big)$.

2.2. *Contact forms*

Let us consider the following 2-forms,

$$\theta^\mu := \mathrm{d}a_j^\mu \wedge \mathrm{d}x^j + 2\sum_{i<j} \tilde{A}_{ij}^\mu \,\mathrm{d}x^i \wedge \mathrm{d}x^j \tag{2.7}$$

defined on $\mathcal{J}(E)$. It is a straightforward matter to verify that the 2-forms (2.7) are invariant geometrical objects on $\mathcal{J}(E)$, henceforth referred to as *contact forms*. The latter allow us to characterize holonomic sections of the fibration $\mathcal{J}(E) \to M$. More precisely, we have the following:

Proposition 2.1. *A section $\gamma : M \to \mathcal{J}(E)$ is holonomic if and only if $\gamma^*(\theta^\mu) = 0$.*

Proof. Given $\gamma : x \to \big(x^i, a_i^\mu(x), \tilde{A}_{ij}^\mu(x)\big)$, for every $\mu = 1, \ldots, r$ one has

$$\begin{aligned}\gamma^*(\theta^\mu) &= \gamma^*\left(\mathrm{d}a_j^\mu \wedge \mathrm{d}x^j + 2\sum_{i<j} \tilde{A}_{ij}^\mu \,\mathrm{d}x^i \wedge \mathrm{d}x^j\right)\\ &= \sum_{i<j}\left[\left(\frac{\partial a_j^\mu(x)}{\partial x^i} - \frac{\partial a_i^\mu(x)}{\partial x^j}\right) + 2\tilde{A}_{ij}^\mu(x)\right]\mathrm{d}x^i \wedge \mathrm{d}x^j.\end{aligned}$$

From this the conclusion follows. □

2.3. *$\mathcal{J}$-prolongation of morphisms*

We show how to define a $\mathcal{J}$-prolongation for a suitable family of bundle morphisms of E

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E \\ \pi\downarrow & & \downarrow\pi \\ M & \xrightarrow{\chi} & M \end{array}$$

projecting to diffeomorphisms of M.

To this end, we first characterize those bundle morphisms (Φ, χ) satisfying the requirement

$$\rho \circ j_1\Phi(w_1) = \rho \circ j_1\Phi(w_2) \qquad \forall\, w_1, w_2 \in \rho^{-1}(z) \tag{2.8}$$

for any $z \in \mathcal{J}(E)$, $j_1\Phi$ denoting the ordinary jet-prolongation of (Φ, χ) on $j_1(E)$.

To start with, supposing that such a bundle morphism (Φ, χ) is described by equations

$$\begin{cases} y^i = \chi^i(x^j) \\ b_i^\nu = \Phi_i^\nu\big(x^j, a_j^\mu\big)\end{cases} \tag{2.9}$$

we recall that its jet-prolongation $j_1\Phi$ on $j_1(E)$ is expressed as (see, for example, [13])

$$\begin{cases} y^i = \chi^i(x^j) \\ b_i^\nu = \Phi_i^\nu\big(x^j, a_j^\mu\big) \\ b_{ij}^\nu = \big(\frac{\partial \Phi_i^\nu}{\partial x^k} + a_{sk}^\mu \frac{\partial \Phi_i^\nu}{\partial a_s^\mu}\big)\big(\frac{\partial (\chi^{-1})^k}{\partial y^j} \circ \chi\big).\end{cases}$$

Condition (2.8) is then mathematically equivalent to requiring that the quantities

$$\frac{1}{2}\left(b^{\nu}_{ij} - b^{\nu}_{ji}\right) = \frac{1}{2}\left[\left(\frac{\partial \Phi^{\nu}_{i}}{\partial x^{k}} + a^{\mu}_{sk}\frac{\partial \Phi^{\nu}_{i}}{\partial a^{\mu}_{s}}\right)\frac{\partial x^{k}}{\partial y^{j}} - \left(\frac{\partial \Phi^{\nu}_{j}}{\partial x^{k}} + a^{\mu}_{sk}\frac{\partial \Phi^{\nu}_{j}}{\partial a^{\mu}_{s}}\right)\frac{\partial x^{k}}{\partial y^{i}}\right] \tag{2.10}$$

depend on antisymmetric combinations of a^{μ}_{sk} only. This is the case if and only if the expressions

$$\left(a^{\mu}_{sk} + a^{\mu}_{ks}\right)\frac{\partial \Phi^{\nu}_{i}}{\partial a^{\mu}_{s}}\frac{\partial x^{k}}{\partial y^{j}} - \left(a^{\mu}_{sk} + a^{\mu}_{ks}\right)\frac{\partial \Phi^{\nu}_{j}}{\partial a^{\mu}_{s}}\frac{\partial x^{k}}{\partial y^{i}}$$

vanish identically. In view of the generality of $\left(a^{\mu}_{sk} + a^{\mu}_{ks}\right)$, this last fact leads to the following equations,

$$\frac{\partial \Phi^{\nu}_{i}}{\partial a^{\mu}_{s}}\frac{\partial x^{k}}{\partial y^{j}} + \frac{\partial \Phi^{\nu}_{i}}{\partial a^{\mu}_{k}}\frac{\partial x^{s}}{\partial y^{j}} - \frac{\partial \Phi^{\nu}_{j}}{\partial a^{\mu}_{s}}\frac{\partial x^{k}}{\partial y^{i}} - \frac{\partial \Phi^{\nu}_{j}}{\partial a^{\mu}_{k}}\frac{\partial x^{s}}{\partial y^{i}} = 0 \tag{2.11}$$

which the bundle morphism (Φ, χ) has to satisfy. In order to solve equations (2.11), let us saturate the indices i and k by $\frac{\partial y^{i}}{\partial x^{k}}$, so obtaining new equations

$$\frac{\partial \Phi^{\nu}_{i}}{\partial a^{\mu}_{k}}\frac{\partial x^{s}}{\partial y^{j}}\frac{\partial y^{i}}{\partial x^{k}} - m\frac{\partial \Phi^{\nu}_{j}}{\partial a^{\mu}_{s}} = 0. \tag{2.12}$$

Once again, saturating equations (2.12) by $\frac{\partial y^{j}}{\partial x^{r}}$, we get the relations

$$\frac{\partial \Phi^{\nu}_{i}}{\partial a^{\mu}_{k}}\frac{\partial y^{i}}{\partial x^{k}}\delta^{s}_{r} = m\frac{\partial \Phi^{\nu}_{j}}{\partial a^{\mu}_{s}}\frac{\partial y^{j}}{\partial x^{r}} \tag{2.13}$$

from which we deduce

$$\frac{\partial \Phi^{\nu}_{j}}{\partial a^{\mu}_{s}}\frac{\partial y^{j}}{\partial x^{r}} = 0 \qquad \text{if} \quad s \neq r \tag{2.14a}$$

and

$$\frac{\partial \Phi^{\nu}_{j}}{\partial a^{\mu}_{s}}\frac{\partial y^{j}}{\partial x^{s}} = \frac{1}{m}\sum_{k=1}^{m}\frac{\partial \Phi^{\nu}_{j}}{\partial a^{\mu}_{k}}\frac{\partial y^{j}}{\partial x^{k}} := \Gamma^{\nu}_{\mu}(x, a) \qquad \forall s \quad (\text{index } s \text{ not repeated}). \tag{2.14b}$$

We may then rewrite equations (2.13) in the form

$$\frac{\partial \Phi^{\nu}_{i}}{\partial a^{\mu}_{r}}\frac{\partial y^{i}}{\partial x^{s}} = \Gamma^{\nu}_{\mu}\delta^{r}_{s} \quad \Leftrightarrow \quad \frac{\partial \Phi^{\nu}_{i}}{\partial a^{\mu}_{r}} = \Gamma^{\nu}_{\mu}\frac{\partial x^{r}}{\partial y^{i}}. \tag{2.15}$$

Moreover, from equations (2.14a) we have

$$0 = \frac{\partial}{\partial a^{\gamma}_{s}}\left(\frac{\partial \Phi^{\nu}_{i}}{\partial a^{\mu}_{r}}\frac{\partial y^{i}}{\partial x^{s}}\right) = \frac{\partial^{2} \Phi^{\nu}_{i}}{\partial a^{\gamma}_{s}\partial a^{\mu}_{r}}\frac{\partial y^{i}}{\partial x^{s}} = \frac{\partial}{\partial a^{\mu}_{r}}\left(\frac{\partial \Phi^{\nu}_{i}}{\partial a^{\gamma}_{s}}\frac{\partial y^{i}}{\partial x^{s}}\right) = \frac{\partial \Gamma^{\nu}_{\gamma}}{\partial a^{\mu}_{r}} \qquad \forall \nu, \gamma, \mu, r$$

showing that the functions Γ^{ν}_{μ} are (pullback of) functions on M, namely $\Gamma^{\nu}_{\mu} = \Gamma^{\nu}_{\mu}(x)$. In view of this, we may solve directly equations (2.15) so getting the final expressions

$$\Phi^{\nu}_{i} = \Gamma^{\nu}_{\mu}(x)\frac{\partial x^{r}}{\partial y^{i}}a^{\mu}_{r} + f^{\nu}_{i}(x) \tag{2.16}$$

where the functions $f^{\nu}_{i}(x) \in \mathcal{F}(M)$ are arbitrary integration terms. Equations (2.16) give the necessary conditions for a bundle morphism (Φ, χ) to satisfy requirement (2.8). Conversely, through a straightforward calculation it is easily seen that the bundle morphisms (2.16) obey the ansatz (2.8). The conclusion follows that equations (2.16) describe the more general bundle morphism (Φ, χ) satisfying (2.8).

For every such bundle morphism (Φ, χ) of E, we may then define the map $\mathcal{J}\Phi : \mathcal{J}(E) \to \mathcal{J}(E)$ expressed as

$$\mathcal{J}\Phi(z) := \rho \circ j_1\Phi(w) \qquad \forall\, w \in \rho^{-1}(z), \quad z \in \mathcal{J}(E) \tag{2.17}$$

according to the commutative diagram

$$\begin{array}{ccc} j_1(E) & \xrightarrow{j_1\Phi} & j_1(E) \\ \rho\big\downarrow & & \big\downarrow\rho \\ \mathcal{J}(E) & \xrightarrow{\mathcal{J}\Phi} & \mathcal{J}(E) \end{array}$$

We shall call the map $\mathcal{J}\Phi$ the *$\mathcal{J}$-prolongation* of (Φ, χ). If the latter is represented by equations (2.9) and (2.16), the explicit expression of $\mathcal{J}\Phi$ is easily deduced from equations (2.10) and it is given by

$$\begin{cases} y^i = \chi^i(x^k) \\ b_i^\nu = \Gamma^\nu_\mu(x)\frac{\partial x^r}{\partial y^i}a_r^\mu + f_i^\nu(x) \\ \tilde{D}^\nu_{ij} = \Gamma^\nu_\mu \tilde{A}^\mu_{ks}\frac{\partial x^k}{\partial y^i}\frac{\partial x^s}{\partial y^j} + \frac{1}{2}\Big[\frac{\partial \Gamma^\nu_\mu}{\partial x^k}\Big(\frac{\partial x^k}{\partial y^j}\frac{\partial x^r}{\partial y^i} - \frac{\partial x^k}{\partial y^i}\frac{\partial x^r}{\partial y^j}\Big)a_r^\mu + \frac{\partial f_i^\nu}{\partial x^k}\frac{\partial x^k}{\partial y^j} - \frac{\partial f_j^\nu}{\partial x^k}\frac{\partial x^k}{\partial y^i}\Big] \end{cases} \tag{2.18}$$

where we have used the notation $\tilde{A}^\mu_{ks} = -\tilde{A}^\mu_{sk}$—henceforth systematically adopted—whenever $k > s$.

Useful characterizations of $\mathcal{J}$-prolongations, similar to those existing for standard jet-prolongations on $j_1(E)$, are given by the following:

Proposition 2.2. *A bundle automorphism* (Ψ, χ) *of* $\mathcal{J}(E) \to M$ *satisfies* $\Psi^*(\eta) \in \mathrm{Span}\{\theta^\sigma, \sigma = 1, \ldots, r\}\, \forall\, \eta \in \mathrm{Span}\{\theta^\sigma, \sigma = 1, \ldots, r\} \Leftrightarrow \Psi = \mathcal{J}\Phi$ *for some bundle automorphism* (Φ, χ) *of* $E \to M$.

Proof. Given (Ψ, χ) of the form

$$\begin{cases} y^i = \chi^i(x^k) \\ b_i^\nu = \Phi_i^\nu\big(x^k, a_k^\mu, \tilde{A}^\mu_{kh}\big) \\ \tilde{D}^\nu_{ij} = \Psi^\nu_{ij}\big(x^k, a_k^\mu, \tilde{A}^\mu_{kh}\big) \end{cases} \tag{2.19}$$

directly from equation (2.7) we have

$$\Psi^*(\theta^\nu) = \left(\frac{\partial \Phi_i^\nu}{\partial x^k}\frac{\partial y^i}{\partial x^s} + \tilde{D}^\nu_{ij}\frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^s}\right)\mathrm{d}x^k \wedge \mathrm{d}x^s + \frac{\partial \Phi_i^\nu}{\partial a_k^\sigma}\frac{\partial y^i}{\partial x^s}\,\mathrm{d}a_k^\sigma \wedge \mathrm{d}x^s$$
$$+ \sum_{p<q}\frac{\partial \Phi_i^\nu}{\partial \tilde{A}^\sigma_{pq}}\frac{\partial y^i}{\partial x^s}\,\mathrm{d}\tilde{A}^\sigma_{pq} \wedge \mathrm{d}x^s.$$

By imposing the requirement $\Psi^*(\theta^\nu) = \Gamma^\nu_\sigma\big(\mathrm{d}a_k^\sigma \wedge \mathrm{d}x^k + \tilde{A}^\sigma_{ks}\,\mathrm{d}x^k \wedge \mathrm{d}x^s\big)$ we get the conditions

$$\frac{\partial \Phi_i^\nu}{\partial \tilde{A}^\sigma_{pq}} = 0 \tag{2.20a}$$

$$\frac{\partial \Phi_i^\nu}{\partial a_k^\sigma}\frac{\partial y^i}{\partial x^s} = \Gamma^\nu_\sigma \delta^k_s \tag{2.20b}$$

$$\tilde{D}^\nu_{ij} = \Gamma^\nu_\sigma \tilde{A}^\sigma_{ks}\frac{\partial x^k}{\partial y^i}\frac{\partial x^s}{\partial y^j} - \frac{1}{2}\left(\frac{\partial \Phi_j^\nu}{\partial x^k}\frac{\partial x^k}{\partial y^i} - \frac{\partial \Phi_i^\nu}{\partial x^k}\frac{\partial x^k}{\partial y^j}\right). \tag{2.20c}$$

From equations (2.20*a*) we deduce that (Ψ, χ) projects to an automorphism (Φ, χ) of $E \to M$. More in particular, since equations (2.20*b*) are identical to the already solved equations (2.15), we derive the explicit expression for the 'vertical' part of (Φ, χ) given by $\Phi^\nu_i(x, a) = \Gamma^\nu_\mu(x) \frac{\partial x^r}{\partial y^i} a^\mu_r + f^\nu_i(x)$. This means that (Φ, χ) is $\mathcal{J}$-prolongable and, under this circumstance, comparison of equations (2.20*c*) with equations (2.18) shows that $\Psi = \mathcal{J}\Phi$.

Conversely, it is a straightforward matter to check that $\mathcal{J}$-prolongations preserve contact forms. □

Proposition 2.3. *Given a bundle automorphism* (Ψ, χ) *of* $\mathcal{J}(E) \to M$, *one has* $\Psi \circ \mathcal{J}\sigma \circ \chi^{-1}$ *is a* $\mathcal{J}$*-extension for every section* $\sigma : E \to M \Leftrightarrow \Psi = \mathcal{J}\Phi$ *for some bundle automorphism* (Φ, χ) *of* $E \to M$.

Proof. ($\Leftarrow$) If $\Psi = \mathcal{J}\Phi$, then in view of propositions 2.1 and 2.2 we have $(\mathcal{J}\Phi \circ \mathcal{J}\sigma \circ \chi^{-1})^* (\theta^\mu) = \chi^{-1*} \circ \mathcal{J}\sigma^* \circ \mathcal{J}\Phi^*(\theta^\mu) = 0 \; \forall \mu = 1, \dots, r$; so, still due to proposition 2.2, $\Psi \circ \mathcal{J}\sigma \circ \chi^{-1}$ is a $\mathcal{J}$-extension.

($\Rightarrow$) Let (Ψ, χ) be of the form (2.19) and let $\sigma(x) = \left(x^i, \sigma^\mu_i(x)\right)$ be a local section of $E \to M$, defined on a open set $U \subset M$. By imposing $\Psi \circ \mathcal{J}\sigma \circ \chi^{-1} = \mathcal{J}\gamma$ for some section $\gamma(y) = \left(y^i, \gamma^\mu_i(y)\right)$ defined on $V = \chi(U)$, we get the relations

$$\gamma^\mu_i(y) = \Phi^\mu_i \left(x^h(y), \sigma^\nu_h(x(y)), \frac{1}{2} \left(\frac{\partial \sigma^\nu_h}{\partial x^k}(x(y)) - \frac{\partial \sigma^\nu_k}{\partial x^h}(x(y)) \right) \right)$$

$$\frac{1}{2} \left(\frac{\partial \gamma^\mu_i}{\partial y^j}(y) - \frac{\partial \gamma^\mu_j}{\partial y^i}(y) \right) = \Psi^\mu_{ij} \left(x^h(y), \sigma^\nu_h(x(y)), \frac{1}{2} \left(\frac{\partial \sigma^\nu_h}{\partial x^k}(x(y)) - \frac{\partial \sigma^\nu_k}{\partial x^h}(x(y)) \right) \right).$$

From these, by a direct calculation we end up with the equations

$$\begin{aligned} \frac{1}{2} \Bigg[\frac{\partial \Phi^\mu_i}{\partial x^h} \frac{\partial x^h}{\partial y^j} + \frac{\partial \Phi^\mu_i}{\partial a^\nu_h} \frac{\partial \sigma^\nu_h}{\partial x^q} \frac{\partial x^q}{\partial y^j} + \sum_{h<k} \frac{\partial \Phi^\mu_i}{\partial \tilde{A}^\nu_{hk}} \frac{1}{2} \left(\frac{\partial^2 \sigma^\nu_h}{\partial x^q \partial x^k}(x(y)) \right. \\ \left. - \frac{\partial^2 \sigma^\nu_k}{\partial x^q \partial x^h}(x(y)) \right) \frac{\partial x^q}{\partial y^j} - \frac{\partial \Phi^\mu_j}{\partial x^h} \frac{\partial x^h}{\partial y^i} - \frac{\partial \Phi^\mu_j}{\partial a^\nu_h} \frac{\partial \sigma^\nu_h}{\partial x^q} \frac{\partial x^q}{\partial y^i} \\ - \sum_{h<k} \frac{\partial \Phi^\mu_j}{\partial \tilde{A}^\nu_{hk}} \frac{1}{2} \left(\frac{\partial^2 \sigma^\nu_h}{\partial x^q \partial x^k}(x(y)) - \frac{\partial^2 \sigma^\nu_k}{\partial x^q \partial x^h}(x(y)) \right) \frac{\partial x^q}{\partial y^i} \Bigg] \\ = \Psi^\mu_{ij} \left(x^h(y), \sigma^\nu_h(x(y)), \frac{1}{2} \left(\frac{\partial \sigma^\nu_h}{\partial x^k}(x(y)) - \frac{\partial \sigma^\nu_k}{\partial x^h}(x(y)) \right) \right). \end{aligned} \tag{2.21}$$

Now, let $\hat{\sigma}(x) = \left(x^i, \hat{\sigma}^\nu_i(x)\right)$ be a second section satisfying the conditions $\frac{\partial \sigma^\nu_i}{\partial x^j}(\tilde{x}) = \frac{\partial \hat{\sigma}^\nu_i}{\partial x^j}(\tilde{x})$ and $\frac{\partial^2 \hat{\sigma}^\nu_i}{\partial x^h \partial x^k}(\tilde{x}) = 0$ at a given $\tilde{x} \in U$. In $\tilde{y} = \chi(\tilde{x})$ we have then

$$\Psi^\mu_{ij}(\mathcal{J}\sigma(\tilde{x}(\tilde{y}))) = \Psi^\mu_{ij}(\mathcal{J}\hat{\sigma}(\tilde{x}(\tilde{y}))).$$

Hence, taking equations (2.21) into account, by subtraction we derive the further equations

$$\begin{aligned} \sum_{h<k} \frac{\partial \Phi^\mu_i}{\partial \tilde{A}^\nu_{hk}} \frac{1}{2} \left(\frac{\partial^2 \sigma^\nu_h}{\partial x^q \partial x^k}(\tilde{x}(\tilde{y})) - \frac{\partial^2 \sigma^\mu_k}{\partial x^q \partial x^h}(\tilde{x}(\tilde{y})) \right) \frac{\partial x^q}{\partial y^j}(\tilde{y}) \\ - \sum_{h<k} \frac{\partial \Phi^\mu_j}{\partial \tilde{A}^\nu_{hk}} \frac{1}{2} \left(\frac{\partial^2 \sigma^\nu_h}{\partial x^q \partial x^k}(\tilde{x}(\tilde{y})) - \frac{\partial^2 \sigma^\nu_k}{\partial x^q \partial x^h}(\tilde{x}(\tilde{y})) \right) \frac{\partial x^q}{\partial y^i}(\tilde{y}) = 0. \end{aligned} \tag{2.22}$$

By choosing sections σ such that $\left(\frac{\partial^2 \sigma^\nu_h}{\partial x^q \partial x^k}(\tilde{x}(\tilde{y})) - \frac{\partial^2 \sigma^\nu_k}{\partial x^q \partial x^h}(\tilde{x}(\tilde{y}))\right) \frac{\partial x^q}{\partial y^j}(\tilde{y}) = \delta^r_h \delta^k_s \delta^\nu_\lambda \delta^j_p$, equation (2.22) implies

$$\frac{\partial \Phi^\mu_i}{\partial \tilde{A}^\lambda_{rs}}(\mathcal{J}\sigma(\tilde{x})) = 0. \tag{2.23}$$

Due to the arbitrariness of σ, equations (2.23) ensure that $\Phi_i^\mu\left(x^j, a_j^\nu\right) \in \mathcal{F}(E)$, namely that (Ψ, χ) is fibred over a bundle morphism (Φ, χ) of $E \to M$. Finally, by a comparison of equations (2.21) with equations (2.10), we conclude that equations (2.21) define a bundle morphism of $\mathcal{J}(E) \to M$ if and only if (Φ, χ) is $\mathcal{J}$-prolongable; in this circumstance, still equations (2.21) imply $\Psi = \mathcal{J}\Phi$. □

2.4. $\mathcal{J}$-prolongation of vector fields

The above stated results allow us to define a $\mathcal{J}$-prolongation for certain vector fields on E projecting to M.

To start with, let $X \in D^1(E)$ be a vector field such that, for each value of the 'time' parameter, its flow is composed of $\mathcal{J}$-prolongable bundle automorphisms of $E \to M$. The $\mathcal{J}$-prolongations of these automorphisms yield a flow on $\mathcal{J}(E)$; the latter may then be differentiated with respect to the parameter, thus providing a vector field $\mathcal{J}(X)$ on $\mathcal{J}(E)$. Taking equation (2.16) into account, it is easily seen that every such a vector field X necessarily needs to be of the form

$$X = \epsilon^i(x^j)\frac{\partial}{\partial x^i} + \left(-\frac{\partial \epsilon^k}{\partial x^q}a_k^\mu + D_\nu^\mu(x^j)a_q^\nu + G_q^\mu(x^j)\right)\frac{\partial}{\partial a_q^\mu} \tag{2.24}$$

where $\epsilon^i(x) = \frac{\mathrm{d}}{\mathrm{d}t}\chi^i\big|_{t=0}$, $D_\nu^\mu(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Gamma_\nu^\mu\big|_{t=0}$, $G_q^\mu(x) = \frac{\mathrm{d}}{\mathrm{d}t}f_\nu^\mu\big|_{t=0}$ are arbitrary functions on M. Moreover, by differentiating equation (2.18) we end up with the local expression

$$\mathcal{J}(X) = \epsilon^i(x^j)\frac{\partial}{\partial x^i} + \left(-\frac{\partial \epsilon^k}{\partial x^q}a_k^\mu + D_\nu^\mu(x^j)a_q^\nu + G_q^\mu(x^j)\right)\frac{\partial}{\partial a_q^\mu} + \sum_{i<j}\tilde{h}_{ij}^\mu\frac{\partial}{\partial \tilde{A}_{ij}^\mu} \tag{2.25a}$$

where

$$\tilde{h}_{ij}^\mu = \frac{1}{2}\left(\frac{\partial D_\nu^\mu}{\partial x^j}a_i^\nu - \frac{\partial D_\nu^\mu}{\partial x^i}a_j^\nu + \frac{\partial G_i^\mu}{\partial x^j} - \frac{\partial G_j^\mu}{\partial x^i}\right) + D_\nu^\mu\tilde{A}_{ij}^\nu + \left(\tilde{A}_{ki}^\mu\frac{\partial \epsilon^k}{\partial x^j} - \tilde{A}_{kj}^\mu\frac{\partial \epsilon^k}{\partial x^i}\right). \tag{2.25b}$$

The above described procedure corresponds essentially to take the standard first jet-prolongation $j_1(X)$ and to project it on $\mathcal{J}(E)$ according to the ansatz

$$\mathcal{J}(X)(\zeta) := \rho_{*\rho^{-1}(\zeta)}(j_1(X)) \qquad \forall\,\zeta \in \mathcal{J}(E). \tag{2.26}$$

The stated assumption on the flow of the vector fields (2.24) ensures that the whole operation is well defined.

In this connection, one could ask which are the most general vector fields X on E, projecting to M, whose first jet-prolongations $j_1(X)$ on $j_1(E)$ pass to the quotient, thus admitting $\mathcal{J}$-prolongations defined without ambiguity by equation (2.26). As it is well known, a necessary and sufficient condition for this to happen is that the requirement

$$[V, j_1(X)] \in \mathcal{D} \qquad \forall\,V \in \mathcal{D} \tag{2.27}$$

holds. Once again, $\mathcal{D}$ denotes the distribution formed by the totality of vertical vectors $V_{pq}^\sigma\frac{\partial}{\partial a_{pq}^\sigma}$ on $j_1(E)$, satisfying the condition $V_{pq}^\sigma = V_{qp}^\sigma$. In this respect, in the appendix we show that condition (2.27) is satisfied by vector fields (2.24) only.

As happens for standard jet-prolongations in ordinary jet-bundles [13], vector fields (2.25) are characterized by preserving contact forms. More specifically, we have the following:

Proposition 2.4. *Let $\pi : \mathcal{J}(E) \to E$ denote the natural projection. Given a vector field Y on $\mathcal{J}(E)$, projectable on E, such that its projection $X(e) := \pi_{*\pi^{-1}(e)}(Y)$ $(\forall\,e \in E)$ defines a vector field on E of the form (2.24), then*

$$Y = \mathcal{J}(X) \quad \Leftrightarrow \quad L_Y\theta^\mu \in \mathrm{Span}\{\theta^\sigma, \sigma = 1, \ldots, r\}.$$

Proof. ($\Leftarrow$) First of all, the immersion (2.5) allows us to represent the contact forms θ^μ as the pullbacks $\theta^\mu = i^*(\hat\theta^\mu)$, $\hat\theta^\mu := \mathrm{d}a^\mu_i \wedge \mathrm{d}x^i + A^\mu_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$ being invariant 2-forms on $(T^*M \otimes \mathcal{G}) \times_M ((T^*M \otimes T^*M) \otimes \mathcal{G})$. We also note that the pullback i^* is an isomorphism between semibasic (with respect to the fibration over E) forms on $\mathcal{J}(E)$ and semibasic forms on $i(\mathcal{J}(E)) \subset (T^*M \otimes \mathcal{G}) \times_M ((T^*M \otimes T^*M) \otimes \mathcal{G})$. Therefore, the Lie derivative $L_Y\theta^\mu$ may be more easily performed working on $(T^*M \otimes \mathcal{G}) \times_M ((T^*M \otimes T^*M) \otimes \mathcal{G})$ and pulling back all the results on $\mathcal{J}(E)$ at the end. More precisely, given a vector field Y as in the hypothesis, let us consider any vector field $\hat Y = \epsilon^i(x^j)\frac{\partial}{\partial x^i} + \left(-\frac{\partial\epsilon^k}{\partial x^q}a^\mu_k + D^\mu_\nu(x^j)a^\nu_q + G^\mu_q(x^j)\right)\frac{\partial}{\partial a^\mu_q} + h^\mu_{ij}\frac{\partial}{\partial A^\mu_{ij}}$ defined on a neighbourhood of $i(\mathcal{J}(E))$ and i_*-related to Y: we have then $L_Y\theta^\mu = i^*(L_{\hat Y}\hat\theta^\mu)$. In view of this, by a direct calculation it is easily seen that the condition $i^*(L_{\hat Y}\hat\theta^\mu) \in \mathrm{Span}\{\theta^\sigma, \sigma = 1, \ldots, r\}$ is mathematically equivalent to the equations

$$L_Y\theta^\mu = D^\mu_\nu\theta^\nu \tag{2.28a}$$

and

$$h^\mu_{ij} + \frac{1}{2}\left(\frac{\partial D^\mu_\nu}{\partial x^i}a^\nu_j - \frac{\partial D^\mu_\nu}{\partial x^j}a^\nu_i + \frac{\partial G^\mu_j}{\partial x^i} - \frac{\partial G^\mu_i}{\partial x^j}\right) + \left(A^\mu_{kj}\frac{\partial\epsilon^k}{\partial x^i} - A^\mu_{ki}\frac{\partial\epsilon^k}{\partial x^j}\right) = D^\mu_\nu A^\nu_{ij}. \tag{2.28b}$$

Comparison with equation (2.25) shows that the only solutions of equations (2.28) are the $\mathcal{J}$-prolongations (2.25).

($\Rightarrow$) It follows directly from proposition 2.2. □

As a consequence we have

Corollary 2.1. *The $\mathcal{J}$-prolongations (2.26) form a Lie algebra.*

Proof. It is a direct consequence of the well-known property of the Lie derivative

$$L_{[\mathcal{J}(X),\mathcal{J}(Y)]}\theta^\mu = L_{\mathcal{J}(X)}L_{\mathcal{J}(Y)}\theta^\mu - L_{\mathcal{J}(Y)}L_{\mathcal{J}(X)}\theta^\mu$$

and of the fact that the vector fields (2.24) form a Lie algebra. □

3. Poincaré–Cartan form and field equations

The aim of this section is to deduce the evolution equations for Yang–Mills theories from a variational principle built on $\mathcal{J}(E)$.

The first step in this direction is to define a suitable Poincaré–Cartan m-form on $\mathcal{J}(E)$. To this end, let

$$L = \mathcal{L}\left(x^i, a^\mu_i, a^\mu_{ij}\right)\mathrm{d}s \tag{3.1}$$

be a Yang–Mills Lagrangian, with $\mathrm{d}s := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$. We recall that the usual Yang–Mills Lagrangian

$$L = -\tfrac{1}{4}F^\mu_{ij}F^{ij}_\mu\sqrt{g}\,\mathrm{d}s \tag{3.2}$$

(with $g := |\det g_{ij}|$) depends only on the antisymmetric part of field derivatives and eventually on non-Abelian terms, not involving field derivatives. Then, we shall consider in the following only Lagrangian densities satisfying this requirement, namely

$$\mathcal{L}\left(x^i, a^\mu_i, a^\mu_{ij}\right) = \hat{\mathcal{L}}\left(x^i, a^\mu_i, A^\mu_{ij} = \tfrac{1}{2}\left(a^\mu_{ij} - a^\mu_{ji}\right)\right) \tag{3.3}$$

where we think $\hat{\mathcal{L}}\left(x^i, a^\mu_i, A^\mu_{ij}\right) \in \mathcal{F}((T^*M \otimes \mathcal{G}) \times_M ((T^*M \otimes T^*M) \otimes \mathcal{G}))$.

We note that, due to the singularity condition

$$\frac{\partial^2 \mathcal{L}}{\partial a^\mu_{ij} \partial a^\sigma_{pq}} V^\sigma_{pq} = 0 \quad \Leftrightarrow \quad V^\sigma_{pq} = V^\sigma_{qp} \tag{3.4}$$

the variational problem built on $j_1(E)$ through the usual Poincaré–Cartan form

$$\Theta_L := \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial a^\mu_{ij}} a^\mu_{ij} \right) \mathrm{d}s + \frac{\partial \mathcal{L}}{\partial a^\mu_{ij}} \mathrm{d}a^\mu_i \wedge \mathrm{d}s_j \tag{3.5}$$

associated with any Lagrangian (3.1), (3.3) (where $\mathrm{d}s_j := \frac{\partial}{\partial x^j} \rfloor \mathrm{d}s$), is not able to ensure that its solutions are automatically jet-extensions. Indeed, as is well known, the latter is a condition which has to be imposed *a priori*.

The problem may be bypassed working directly on $\mathcal{J}(E)$ where, as we shall see, the kinematic admissibility of the critical sections is granted by the variational problem itself.

To start with, taking equation (3.4) as well as the verticality of the distribution $\mathcal{D}$ into account, one may easily prove the relations

$$\mathcal{D} \rfloor \Theta_L = 0 \qquad \mathcal{D} \rfloor \mathrm{d}\Theta_L = 0. \tag{3.6}$$

Equations (3.6) ensure that the Poincaré–Cartan form Θ_L passes to the quotient, namely that there exists an m-form $\tilde{\Theta}_L$ on $\mathcal{J}(E)$ such that $\Theta_L = \rho^*(\tilde{\Theta}_L)$ (see, for example, [14, 15]).

In local coordinates, recalling immersion (2.5) and taking the straightforward identities $\frac{\partial \mathcal{L}}{\partial a^\mu_{ij}} = \frac{1}{2} \frac{\partial \hat{\mathcal{L}}}{\partial A^\mu_{pq}} \delta^{ij}_{pq}$ and $\frac{\partial \mathcal{L}}{\partial a^\mu_{ij}} a^\mu_{ij} = \frac{1}{2} \frac{\partial \hat{\mathcal{L}}}{\partial A^\mu_{pq}} \delta^{ij}_{pq} A^\mu_{ij}$ into account, it is easily seen that $\tilde{\Theta}_L = i^*(\hat{\theta}_L)$,

$$\hat{\Theta}_L = \left(\hat{\mathcal{L}} - \frac{1}{2} \frac{\partial \hat{\mathcal{L}}}{\partial A^\mu_{ij}} \delta^{pq}_{ij} A^\mu_{pq} \right) \mathrm{d}s + \frac{1}{2} \frac{\partial \hat{\mathcal{L}}}{\partial A^\mu_{ij}} \delta^{pq}_{ij} \mathrm{d}a^\mu_p \wedge \mathrm{d}s_q \tag{3.7}$$

being a suitable m-form on $(T^*M \otimes \mathcal{G}) \times_M ((T^*M \otimes T^*M) \otimes \mathcal{G})$. Introducing the pullback $\tilde{\mathcal{L}} := i^*(\hat{\mathcal{L}})$, one has the relations $\frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{A}^\mu_{ij}} = \frac{\partial \hat{\mathcal{L}}}{\partial A^\mu_{ij}} - \frac{\partial \hat{\mathcal{L}}}{\partial A^\mu_{ji}}$ for $i < j$. Then, setting $\frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{A}^\mu_{ij}} := -\frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{A}^\mu_{ji}}$ and $\tilde{A}^\mu_{ij} := -\tilde{A}^\mu_{ji}$ for $i > j$, directly from equation (3.7) we derive the local expression[5]

$$\tilde{\Theta}_L = \left(\tilde{\mathcal{L}} - \frac{1}{2} \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{A}^\mu_{ij}} \tilde{A}^\mu_{ij} \right) \mathrm{d}s + \frac{1}{2} \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{A}^\mu_{ij}} \mathrm{d}a^\mu_i \wedge \mathrm{d}s_j. \tag{3.8}$$

It is now convenient to introduce new coordinates which are more suitable for the formalism of connections. To this end, given a connection 1-form $a^\mu(x) = a^\mu_i(x)\, \mathrm{d}x^i$, we recall that its curvature is defined as

$$F^\mu = \tfrac{1}{2} F^\mu_{ji}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j = \mathrm{d}a^\mu(x) - \tfrac{1}{2} a^\nu(x) \wedge a^\rho(x) C^\mu_{\rho\nu} \tag{3.9}$$

where $C^\mu_{\rho\nu}$ are the structure coefficients of the Lie algebra $\mathcal{G}$. Equation (3.9) suggests taking the components of the curvature as coordinates on the fibres of $\mathcal{J}(E) \to E$; they are related to the $\mathcal{J}$-coordinates by the following transformation laws

$$x^i = x^i \qquad a^\mu_i = a^\mu_i \qquad F^\mu_{ji} = -2\tilde{A}^\mu_{ij} - a^\nu_i a^\rho_j C^\mu_{\rho\nu} \qquad (i < j). \tag{3.10}$$

In connection with this, we also define the following forms on $\mathcal{J}(E)$:

$$F^\mu := \tfrac{1}{2} F^\mu_{ji}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \qquad \left(\text{with } F^\mu_{ji} := -F^\mu_{ij} \text{ if } i > j\right) \tag{3.11a}$$

$$\Omega^\mu_i := \mathrm{d}a^\mu_i + \tfrac{1}{2} a^\nu_i C^\mu_{\rho\nu} a^\rho_j\, \mathrm{d}x^j \tag{3.11b}$$

$$\Omega^\mu := -\mathrm{d}x^i \wedge \Omega^\mu_i. \tag{3.11c}$$

[5] In equation (3.8) the indices i and j in the summations run from 1 to m.

As a result, it is easily seen that the Poincaré–Cartan form (3.8) may be rewritten as

$$\tilde{\Theta}_L = \tilde{\mathcal{L}}\,\mathrm{d}s - \frac{1}{2}\frac{\partial\tilde{\mathcal{L}}}{\partial F^\mu_{ji}} F^\mu_{ji}\,\mathrm{d}s - \frac{\partial\tilde{\mathcal{L}}}{\partial F^\mu_{ji}}\Omega^\mu_i \wedge \mathrm{d}s_j. \tag{3.12}$$

Moreover, taking the straightforward identities $\frac{\partial\tilde{\mathcal{L}}}{\partial F^\mu_{ji}} F^\mu_{ji}\,\mathrm{d}s = F^\mu \wedge \frac{\partial\tilde{\mathcal{L}}}{\partial F^\mu_{ji}}\,\mathrm{d}s_{ji}$ (with $\mathrm{d}s_{ji} := \frac{\partial}{\partial x^j} \rfloor \frac{\partial}{\partial x^i} \rfloor \mathrm{d}s$), $\frac{\partial\tilde{\mathcal{L}}}{\partial F^\mu_{ji}}\Omega^\mu_i \wedge \mathrm{d}s_j = -\frac{1}{2}\Omega^\mu \wedge \frac{\partial\tilde{\mathcal{L}}}{\partial F^\mu_{ji}}\,\mathrm{d}s_{ji}$ and $\theta^\mu = \Omega^\mu - F^\mu$ into account, we get the final expression

$$\tilde{\Theta}_L = \tilde{\mathcal{L}}\,\mathrm{d}s + \tfrac{1}{2}\theta^\mu \wedge P_\mu \tag{3.13}$$

where $P_\mu := \frac{\partial\tilde{\mathcal{L}}}{\partial F^\mu_{ji}}\,\mathrm{d}s_{ji}$.

In what follows, for any compact domain $D \subset M$ we shall denote by Γ_D the totality of sections $\gamma : U \subset M \to \mathcal{J}(E)$ defined in some open neighbourhood $U \supset D$.

By means of the *m*-form $\tilde{\Theta}_L$ we may then introduce a real-valued action functional A_D on Γ_D, expressed as

$$A_D(\gamma) := \int_D \gamma^*(\tilde{\Theta}_L). \tag{3.14}$$

Given a vertical (with respect to the fibration $\mathcal{J}(E) \to M$) vector field $X = X^\mu_i \frac{\partial}{\partial a^\mu_i} + \sum_{i<j} X^\mu_{ji} \frac{\partial}{\partial F^\mu_{ji}}$ on $\mathcal{J}(E)$ and a section $\gamma \in \Gamma_D$ we may construct sections $\gamma_\xi := \Phi_\xi \circ \gamma \in \Gamma_D$ by dragging γ along the flow Φ_ξ of *X*.

The *first variation* of A_D at γ in the direction *X* is defined as

$$\frac{\delta A_D}{\delta X}(\gamma) := \frac{\mathrm{d}}{\mathrm{d}\xi}\int_D \gamma^*_\xi(\tilde{\Theta}_L)\bigg|_{\xi=0} = \int_D \gamma^*(X \rfloor \mathrm{d}\tilde{\Theta}_L) + \int_{\partial D} \gamma^*(X \rfloor \tilde{\Theta}_L). \tag{3.15}$$

A section $\gamma : M \to \mathcal{J}(E)$ is called *critical* if $\frac{\delta A_D}{\delta X}(\gamma) = 0$ for all compact domains $D \subset M$ and all vertical vector fields vanishing on the boundary ∂D. From equation (3.15), taking the condition at the boundary into account, it follows that $\gamma(x^k) = \left(x^k, a^\mu_i(x^k), F^\mu_{ji}(x^k)\right)$ is critical if and only if it satisfies the equation

$$\gamma^*(X \rfloor \mathrm{d}\tilde{\Theta}_L) = 0 \qquad \forall\, X = X^\mu_i \frac{\partial}{\partial a^\mu_i} + \sum_{i<j} X^\mu_{ji}\frac{\partial}{\partial F^\mu_{ji}} \qquad X|_{\partial D} = 0. \tag{3.16}$$

It is now convenient to introduce the following local dual bases of $T\mathcal{J}(E)$ and $T^*\mathcal{J}(E)$, respectively expressed as

$$\begin{aligned} \frac{D}{Dx^k} &:= \frac{\partial}{\partial x^k} + \frac{1}{2}a^\lambda_k a^\beta_i C^\mu_{\beta\lambda}\frac{\partial}{\partial a^\mu_i} - \sum_{r>s} F^\lambda_{rs} a^\rho_k C^\mu_{\rho\lambda}\frac{\partial}{\partial F^\mu_{rs}} \\ \frac{D}{Da^\mu_i} &:= \frac{\partial}{\partial a^\mu_i}, \qquad \frac{D}{DF^\mu_{ji}} := \frac{\partial}{\partial F^\mu_{ji}} \end{aligned} \tag{3.17a}$$

and

$$Dx^i := \mathrm{d}x^i, \qquad \Omega^\mu_i, \qquad DF^\mu_{ji} := \mathrm{d}F^\mu_{ji} + F^\lambda_{ji} a^\rho C^\mu_{\rho\lambda} \tag{3.17b}$$

The latter are consistent with the definition of the covariant differential operator *D*, acting on tensorial *p*-forms on $\mathcal{J}(E)$ of adjoint and co-adjoint kind, respectively, as

$$D\eta^\mu = \mathrm{d}\eta^\mu + (-1)^p \eta^\lambda a^\rho C^\mu_{\rho\lambda} \tag{3.18a}$$

$$D\eta_\mu = \mathrm{d}\eta_\mu - (-1)^p \eta_\lambda a^\rho C^\lambda_{\rho\mu}. \tag{3.18b}$$

Using these latter and taking the identity $\frac{\partial\tilde{\mathcal{L}}}{\partial F^\mu_{ji}}\,\mathrm{d}s = \frac{1}{2}\,\mathrm{d}x^i \wedge \mathrm{d}x^j P_\mu$ into account, it is easily seen that

$$\mathrm{d}\Theta = D\Theta = \frac{D\tilde{\mathcal{L}}}{Da^\mu_i}\Omega^\mu_i \wedge \mathrm{d}s + \frac{1}{2}\theta^\mu \wedge DP_\mu. \tag{3.19}$$

Then, given an arbitrary vector field $X = b^\mu_i \frac{D}{Da^\mu_i} + \sum_{r>s} h^\mu_{rs}\frac{D}{DF^\mu_{rs}}$ as in equation (3.16), from equation (3.19) we have

$$X \rfloor \mathrm{d}\Theta = X \rfloor D\Theta = b^\mu_i\left[\frac{D\tilde{\mathcal{L}}}{Da^\mu_i}\,\mathrm{d}s + \frac{1}{2}\,\mathrm{d}x^i \wedge DP_\mu\right] + \frac{1}{2}\theta^\mu \wedge \left[b^\lambda_k \frac{D^2\mathcal{L}}{Da^\lambda_k DF^\mu_{ji}} + \sum_{r>s} h^\lambda_{rs}\frac{D^2\mathcal{L}}{DF^\lambda_{rs}DF^\mu_{ji}}\right]\mathrm{d}s_{ji}. \tag{3.20}$$

Due to the arbitrariness of X, by imposing the requirement $\gamma^*(X \rfloor \mathrm{d}\Theta) = 0$, we finally obtain two sets of equations. The first one is given by

$$\sum_{p<q}\gamma^*\left(\frac{\partial^2\tilde{\mathcal{L}}}{\partial F^\mu_{ji}\partial F^\sigma_{qp}}\right)\left(2\tilde{A}^\sigma_{pq}(x) - \delta^{rs}_{pq}\frac{\partial a^\sigma_r(x)}{\partial x^s}\right) = 0 \qquad \forall\, i < j. \tag{3.21a}$$

If $\tilde{\mathcal{L}}$ is a Yang–Mills Lagrangian, then equation (3.21*a*) ensures that the critical section γ is holonomic, i.e. $\tilde{A}^\mu_{ij}(x) = \frac{1}{2}\big(\frac{\partial a^\mu_i(x)}{\partial x^j} - \frac{\partial a^\mu_j(x)}{\partial x^i}\big)$.

This last result allows us to write the second set of equations yielded by equation (3.16) in the form

$$\gamma^*\left(\frac{\partial\tilde{\mathcal{L}}}{\partial a^\mu_i} + D_j\frac{\partial\tilde{\mathcal{L}}}{\partial F^\mu_{ji}}\right) = 0. \tag{3.21b}$$

The latter are the field equations of the problem we are studying. In this respect it is easy to verify that equations (3.21*b*) are exactly the Euler–Lagrange equations induced by the Lagrangian (3.1).

It is worth noting that the restriction on the verticality of the vector fields X in equation (3.16) may be removed. In fact, it is a straightforward matter to see that equation (3.16) implies automatically $\gamma^*(X \rfloor \mathrm{d}\tilde{\Theta}_L) = 0\ \forall\, X \in D^1(\mathcal{J}(E))$.

4. Symmetries and the Nöther theorem

In this section, we briefly investigate the relationships between symmetries and the Nöther theorem in the present geometrical framework.

We begin by stating

Definition 4.1. *A vector field Z on $\mathcal{J}(E)$ is called a* generalized infinitesimal Lagrangian symmetry *if it satisfies the requirement*

$$L_Z(\tilde{\mathcal{L}}\,\mathrm{d}s) = \mathrm{d}\alpha \tag{4.1}$$

for some $(m-1)$-form α on $\mathcal{J}(E)$.

Of course, equation (4.1) considers the trivial case $L_Z(\tilde{\mathcal{L}}\,\mathrm{d}s) = 0$. In such a circumstance, if Z is projectable to M and (Ψ_s, χ_s) denotes its flow, then the condition $L_Z(\tilde{\mathcal{L}}\,\mathrm{d}s) = 0$ locally reads

$$\tilde{\mathcal{L}} = \det\left|\frac{\partial\chi^i_s}{\partial x^j}\right|\tilde{\mathcal{L}}\circ\Psi_s \qquad \forall\, s \tag{4.2}$$

yielding the well-known behaviour of the Lagrangian density $\tilde{\mathcal{L}}$ under the action of usual Lagrangian symmetries.

Also, borrowing from [12], we introduce the following:

Definition 4.2. *A vector field Z on $\mathcal{J}(E)$ is called a* Nöther vector field *if it satisfies the following condition,*

$$L_Z\tilde{\Theta}_L = \omega + \mathrm{d}\alpha \tag{4.3}$$

where ω is an m-form belonging to the ideal generated by the forms θ^σ and α is a generic $(m-1)$-form on $\mathcal{J}(E)$.

Once again, equation (4.3) considers the case $L_Z\tilde{\Theta}_L = 0$. When this happens and, as above, Z is projectable to M, indicating again by (Ψ_s, χ_s) its flow, it can immediately be seen that $\Psi_s^* \circ \gamma \circ \chi_{-s}^*$ is a critical section if γ is, namely Z is an infinitesimal dynamical symmetry of the theory.

Proposition 4.1. *If a generalized infinitesimal Lagrangian symmetry Z is a $\mathcal{J}$-prolongation of some vector field (2.24) on E, then it is a Nöther vector field.*

Proof. Given such a Z, recalling the representation (3.13) for the Poincaré–Cartan form $\tilde{\Theta}_L$, we have

$$L_Z(\tilde{\Theta}_L) = \mathrm{d}\alpha + \tfrac{1}{2}L_Z(\theta^\mu)\wedge P_\mu + \tfrac{1}{2}\theta^\mu \wedge L_Z(P_\mu). \tag{4.4}$$

Proposition 2.4 ensures that the last two terms on the right-hand side of equation (4.4) belong to the ideal generated by contact forms. From this the conclusion follows. □

Nöther vector fields allow us to restate in the present geometrical framework a generalized Nöther theorem associating with every Nöther vector field Z a conserved current $\mathcal{E}$. In fact, for any Z satisfying equation (4.3) we have

$$\mathrm{d}(Z \rfloor \tilde{\Theta}_L - \alpha) = (\omega - Z \rfloor \mathrm{d}\tilde{\Theta}_L). \tag{4.5}$$

If $\gamma : M \to \mathcal{J}(E)$ is a critical section we have

$$\mathrm{d}\gamma^*(Z \rfloor \tilde{\Theta}_L - \alpha) = \gamma^*(\omega - Z \rfloor \mathrm{d}\tilde{\Theta}_L) = 0. \tag{4.6}$$

As a consequence, denoting by $\mathcal{E} := \gamma^*(Z \rfloor \tilde{\Theta}_L - \alpha)$, we conclude that the *Nöther current* $\mathcal{E}$ is conserved (on shell), i.e. $\mathrm{d}\mathcal{E} = 0$.

Proposition 4.2. *If a Nöther vector field Z is a $\mathcal{J}$-prolongation of some vector field (2.24) on E, then it is an infinitesimal dynamical symmetry.*

Proof. Let us denote by (Ψ_s, χ_s) ($\chi_s : M \to M$ local diffeomorphisms) the flow of Z and by $\gamma_s := \Psi_s \circ \gamma \circ \chi_s^{-1}$ the variation of a critical section γ along Z. Since Z is a $\mathcal{J}$-prolongation, γ_s are automatically $\mathcal{J}$-extensions. Therefore, the holonomy of any γ_s being ensured, we may take first variations with respect to $\mathcal{J}$-prolongations only. Then, let $\mathcal{J}(X)$ be an arbitrary $\mathcal{J}$-prolongation (vertical with respect to the fibration over M) and let Φ_ξ denote its correspondent flow. Without loss of generality, we may assume the deformations of any γ_s associated with $\mathcal{J}(X)$ to be of the form $\gamma_{\xi,s} := \Psi_s \circ \Phi_\xi \circ \gamma \circ \chi_s^{-1}$. We want to prove $\frac{\delta A_{\chi_s(D)}}{\delta \mathcal{J}(X)}(\gamma_s) := \frac{\mathrm{d}}{\mathrm{d}\xi}\left(A_{\chi_s(D)}(\gamma_{\xi,s})\right)\big|_{\xi=0} = 0\ \forall s, \forall \mathcal{J}(X)$ vanishing at ∂D. To this end we note that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\delta A_{\chi_s(D)}}{\delta \mathcal{J}(X)}(\gamma_s)\right)\bigg|_{s=\bar{s}} = \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}s}A_{\chi_s(D)}(\gamma_{\xi,s})\bigg|_{\xi=0,s=\bar{s}} = \frac{\mathrm{d}}{\mathrm{d}\xi}\int_D \gamma^* \circ \Phi_\xi^* \circ \Psi_{\bar{s}}^*(\omega + \mathrm{d}\alpha)\bigg|_{\xi=0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\xi}\int_{\chi_s(D)} \chi_{-\bar{s}}^* \circ \gamma^* \circ \Phi_\xi^* \circ \Psi_{\bar{s}}^*(\omega)\bigg|_{\xi=0} + \frac{\mathrm{d}}{\mathrm{d}\xi}\int_D \gamma^* \circ \Phi_\xi^*\, \mathrm{d}(\Psi_{\bar{s}}^*\alpha)\bigg|_{\xi=0} = 0. \tag{4.7}$$

The first integral vanishes because of the holonomy of the sections $\gamma_{\xi,\bar{s}}$, the second one because $\mathcal{J}(X)$ vanishes on ∂D. Therefore we have

$$\frac{\delta A_{\chi_s(D)}}{\delta \mathcal{J}(X)}(\gamma_s) = \frac{\delta A_D}{\delta \mathcal{J}(X)}(\gamma) = 0 \qquad \forall s. \tag{4.8}$$

We conclude that all γ_s are critical sections and then Z is a dynamical symmetry. □

5. Example

Let us consider the free Yang–Mills Lagrangian[6] on the Minkowski spacetime. Using pseudo-Euclidean coordinates for simplicity, the latter is expressed as

$$L = \tilde{\mathcal{L}}(x, a, F)\,\mathrm{d}s := -\tfrac{1}{4} F^{\mu}_{ij} F^{ij}_{\mu}\,\mathrm{d}s \tag{5.1}$$

where $F^{ij}_{\mu} := F^{\nu}_{pq}\eta^{pi}\eta^{qj}\gamma_{\mu\nu}$. The associated Poincaré–Cartan form (3.13) assumes the expression

$$\tilde{\Theta}_L = -\tfrac{1}{4} F^{\mu}_{ij} F^{ij}_{\mu}\,\mathrm{d}s - \tfrac{1}{2}\theta^{\mu} \wedge F^{ij}_{\mu}\,\mathrm{d}s_{ij}. \tag{5.2}$$

If $\sigma(x) = \big(x, a^{\mu}_i(x), F^{\mu}_{ji}(x)\big)$ denotes a critical section for the variational problem built through $\tilde{\Theta}_L$, then equations (3.21) imply

$$F^{\mu}_{ji}(x) = -\frac{\partial a^{\mu}_i}{\partial x^j}(x) + \frac{\partial a^{\mu}_j}{\partial x^i}(x) - a^{\nu}_i(x) a^{\rho}_j(x) C^{\mu}_{\rho\nu} \qquad \text{and} \qquad D_j F^{ji}_{\mu}(x) = 0. \tag{5.3}$$

The latter are the well-known Euler–Lagrange equations for the free Yang–Mills field.

Infinitesimal gauge transformations may be represented by vector fields X on E of the form

$$X = D_i b^{\mu} \frac{\partial}{\partial a^{\mu}_i} \tag{5.4}$$

where $b^{\mu} = b^{\mu}(x) \in F(M)$ and $D_i b^{\mu} = \frac{\partial b^{\mu}}{\partial x^i} + b^{\nu} a^{\sigma}_i C^{\mu}_{\sigma\nu}$.

Comparison with equation (2.24) shows that the vector fields (5.4) are $\mathcal{J}$-prolongable. From equations (2.25), (3.17*a*) we easily get the representation

$$\mathcal{J}(X) = D_i b^{\mu} \frac{D}{Da^{\mu}_i} + \frac{1}{2} b^{\nu} F^{\rho}_{ij} C^{\mu}_{\rho\nu} \frac{D}{DF^{\mu}_{ij}} \tag{5.5}$$

for the $\mathcal{J}$-prolongation on $\mathcal{J}(E)$ of X.

As is well known, gauge transformations are Lagrangian symmetries. Indeed, in the present geometrical setting we have the identity

$$L_{\mathcal{J}(X)}(\tilde{\mathcal{L}}\,\mathrm{d}s) = L_{\mathcal{J}(X)}\big(-\tfrac{1}{4} F^{\mu}_{ij} F^{ij}_{\mu}\,\mathrm{d}s\big) = -\tfrac{1}{2} b^{\nu} F^{\rho}_{ij} C^{\mu}_{\rho\nu} F^{ij}_{\mu} = 0 \tag{5.6}$$

the vanishing of (5.6) being due to the relation $\gamma_{\mu\rho} C^{\mu}_{\nu\sigma} = \gamma_{\mu[\rho} C^{\mu}_{\nu\sigma]}$, a consequence of the adjoint invariance of the metric γ.

Therefore, from proposition 4.1 it follows that the vector fields (5.5) are Nöther vector fields and so, in view of proposition 4.2, they are infinitesimal dynamical symmetries of the theory.

If σ is a critical section, then we have

$$0 = \sigma^* L_{\mathcal{J}(X)}(\tilde{\Theta}_L) = \mathrm{d}\sigma^*(\mathcal{J}(X) \lrcorner\, \tilde{\Theta}_L) \tag{5.7}$$

[6] The extension to interacting Yang–Mills field theories is presently the object of further studies; preliminary results show that, at least in the case of minimal coupling, the present formalism is perfectly working.

showing that

$$\mathcal{E} = \sigma^*(\mathcal{J}(X) \rfloor \tilde{\Theta}_L) = \sigma^*\big(\tfrac{1}{2} Db^\mu \wedge P_\mu\big) = D_i b^\mu(x) F^{ij}_\mu(x)\, \mathrm{d}s_j \tag{5.8}$$

are the conserved Nöther currents, associated with the fields (5.5).

Another family of Nöther vector fields is obtained by considering lifts to E of Killing vector fields on M.

More precisely, let $K = \xi^i \frac{\partial}{\partial x^i}$ be a Killing vector field on M. The natural lift to E of K is given by

$$Y = \xi^i \frac{\partial}{\partial x^i} - \frac{\partial \xi^j}{\partial x^i} a^\mu_j \frac{\partial}{\partial a^\mu_i}. \tag{5.9}$$

Once again, taking equation (2.24) into account, it is immediately seen that Y is $\mathcal{J}$-prolongable. In local coordinates we have the expression

$$\mathcal{J}(Y) = \xi^i \frac{\partial}{\partial x^i} - \frac{\partial \xi^j}{\partial x^i} a^\mu_j \frac{\partial}{\partial a^\mu_i} + \frac{\partial \xi^k}{\partial x^i} F^\mu_{jk} \frac{\partial}{\partial F^\mu_{ij}}. \tag{5.10}$$

Moreover, it is easy to verify that vector fields (5.10) are infinitesimal Lagrangian symmetries. Indeed one has

$$L_{\mathcal{J}(Y)}(\tilde{\mathcal{L}}\, \mathrm{d}s) = -\frac{\partial \xi^k}{\partial x^i} F^\mu_{jk} F^{ij}_\mu\, \mathrm{d}s + \frac{\partial \xi^k}{\partial x^k}\, \mathrm{d}s = 0 \tag{5.11}$$

because of the identities $\frac{\partial \xi^k}{\partial x^i} + \frac{\partial \xi^i}{\partial x^k} = 0$.

In view of this, as above we have that every field (5.10) is a Nöther vector field and, as in the case of infinitesimal gauge transformations, it represents an infinitesimal dynamical symmetry. From equations (4.3) and (4.6) it follows that the pullback under critical section σ

$$\mathcal{E} = \sigma^*(\mathcal{J}(Y) \rfloor \tilde{\Theta}_L) \tag{5.12}$$

is the corresponding conserved current. In more detail, after a straightforward calculation, we get the expression

$$\begin{aligned}\mathcal{J}(Y) \rfloor \tilde{\Theta}_L = {} & \xi^k\big(-\tfrac{1}{4} F^\mu_{ij} F^{ij}_\mu \delta^h_k + F^\mu_{ik} F^{ih}_\mu\big)\, \mathrm{d}s_h + \mathrm{d}\big(\xi^k a^\mu_k F^{ij}_\mu\, \mathrm{d}s_{ij}\big) \\ & - \xi^k a^\mu_k D\big(F^{ij}_\mu\, \mathrm{d}s_{ij}\big) + \tfrac{1}{2}\theta^\mu \wedge (\mathcal{J}(Y) \rfloor P_\mu).\end{aligned} \tag{5.13}$$

The last two terms on the right-hand side of equation (5.13) clearly vanish on the critical section. Then we have

$$\mathcal{E} = \xi^k(x)\big(-\tfrac{1}{4} F^\mu_{ij}(x) F^{ij}_\mu(x) \delta^h_k + F^\mu_{ik}(x) F^{ih}_\mu(x)\big)\, \mathrm{d}s_h + \mathrm{d}\big(\xi^k(x) a^\mu_k(x) F^{ij}_\mu(x)\, \mathrm{d}s_{ij}\big) \tag{5.14}$$

involving, besides the negligible exact term, the stress–energy tensor

$$T^h_k := -\tfrac{1}{4} F^\mu_{ij} F^{ij}_\mu \delta^h_k + F^\mu_{ik} F^{ih}_\mu \tag{5.15}$$

Appendix. $\mathcal{J}$-prolongability of vector fields

Given a vector field X on E projecting to M of the form

$$X = \epsilon^i(x^r) \frac{\partial}{\partial x^i} + b^\mu_i\big(x^r, a^\sigma_r\big) \frac{\partial}{\partial a^\mu_i} \tag{A.1}$$

let us indicate by

$$j_1(X) = \epsilon^i(x^r) \frac{\partial}{\partial x^i} + b^\mu_i\big(x^r, a^\sigma_r\big) \frac{\partial}{\partial a^\mu_i} + \left(\frac{\partial b^\mu_i}{\partial x^j} + \frac{\partial b^\mu_i}{\partial a^\nu_k} a^\nu_{kj} - a^\mu_{ik} \frac{\partial \epsilon^k}{\partial x^j}\right) \frac{\partial}{\partial a^\mu_{ij}}$$

its first jet-prolongation on $\mathcal{J}(E)$. Denoting by $V = V^\mu_{ij} \frac{\partial}{\partial a^\mu_{ij}}$ an arbitrary vector field belonging to the distribution $\mathcal{D}$, a straightforward calculation shows that condition (2.27) is mathematically equivalent to requiring that the quantity

$$V^\mu_{rk}\left(\delta^k_j \frac{\partial b^\nu_i}{\partial a^\mu_r} - \frac{\partial \epsilon^k}{\partial x^j}\delta^r_i \delta^\nu_\mu\right) \tag{A.2}$$

is symmetric in the indices i and j for all V^μ_{rk} symmetric in the indices r and k; this means that the $[ij]$-skewsymmetric part of expression (A.2) must vanish. This leads to the following equations:

$$\left(\delta^k_j \frac{\partial b^\nu_i}{\partial a^\mu_r} - \frac{\partial \epsilon^k}{\partial x^j}\delta^r_i \delta^\nu_\mu\right) + \left(\delta^r_j \frac{\partial b^\nu_i}{\partial a^\mu_k} - \frac{\partial \epsilon^r}{\partial x^j}\delta^k_i \delta^\nu_\mu\right) - \left(\delta^k_i \frac{\partial b^\nu_j}{\partial a^\mu_r} - \frac{\partial \epsilon^k}{\partial x^i}\delta^r_j \delta^\nu_\mu\right)$$
$$- \left(\delta^r_i \frac{\partial b^\nu_j}{\partial a^\mu_k} - \frac{\partial \epsilon^r}{\partial x^i}\delta^k_j \delta^\nu_\mu\right) = 0.$$

Saturating the indices j and r we get the differential equations

$$\left(\delta^\nu_\mu \frac{\partial \epsilon^r}{\partial x^j} + \frac{\partial b^\nu_j}{\partial a^\mu_r}\right)\left(\delta^k_i \delta^j_r - m\delta^k_r \delta^j_i\right) = 0 \tag{A.3}$$

for the components $\epsilon^i(x^r)$ and $b^\mu_i\left(x^r, a^\sigma_r\right)$ of X. From equation (A.3) we have the following:

- If $k = i$, then
$$\delta^\nu_\mu \frac{\partial \epsilon^k}{\partial x^k} + \frac{\partial b^\nu_k}{\partial a^\mu_k} = \frac{1}{m}\sum_{s=1}^m \left(\delta^\nu_\mu \frac{\partial \epsilon^s}{\partial x^s} + \frac{\partial b^\nu_s}{\partial a^\mu_s}\right) \qquad \text{(index } k \text{ not repeated)}. \tag{A.4}$$
- If $k \neq i$, then
$$\delta^\nu_\mu \frac{\partial \epsilon^k}{\partial x^i} + \frac{\partial b^\nu_i}{\partial a^\mu_k} = 0. \tag{A.5}$$

From equation (A.5) we deduce that

$$b^\nu_i = -\left(\frac{\partial \epsilon^1}{\partial x^i}a^\nu_1 + \frac{\partial \epsilon^2}{\partial x^i}a^\nu_2 + \cdots + \widehat{\frac{\partial \epsilon^i}{\partial x^i}a^\nu_i} + \cdots + \frac{\partial \epsilon^m}{\partial x^i}a^\nu_m\right) + f^\nu_i\left(x^j, a^\lambda_i\right) \tag{A.6}$$

where ν and i are fixed and the symbol $\widehat{}$ denotes the absence of the corresponding term.

Inserting equation (A.6) in equation (A.4) we obtain the following equations,

$$\delta^\nu_\mu \frac{\partial \epsilon^k}{\partial x^k} + \frac{\partial f^\nu_k}{\partial a^\mu_k} = \frac{1}{m}\sum_{s=1}^m \left(\delta^\nu_\mu \frac{\partial \epsilon^s}{\partial x^s} + \frac{\partial f^\nu_s}{\partial a^\mu_s}\right)$$

for the functions f^ν_i. We may single out two different cases:

- If $\nu = \mu$, then
$$\frac{\partial \epsilon^k}{\partial x^k} + \frac{\partial f^\nu_k}{\partial a^\nu_k} = \frac{1}{m}\sum_{s=1}^m \left(\frac{\partial \epsilon^s}{\partial x^s} + \frac{\partial f^\nu_s}{\partial a^\nu_s}\right). \tag{A.7}$$
- If $\nu \neq \mu$, then
$$\frac{\partial f^\nu_k}{\partial a^\mu_k} = \frac{1}{m}\sum_{s=1}^m \left(\frac{\partial f^\nu_s}{\partial a^\mu_s}\right). \tag{A.8}$$

Equation (A.8) yields

$$\frac{\partial f_1^\nu}{\partial a_1^\mu} = \frac{\partial f_2^\nu}{\partial a_2^\mu} = \cdots = \frac{\partial f_m^\nu}{\partial a_m^\mu} \tag{A.9}$$

whenever $\nu \neq \mu$. From equation (A.9), taking second derivatives into account, we get

$$\frac{\partial^2 f_i^\nu}{\partial a_i^\lambda \partial a_i^\mu} = \frac{\partial^2 f_j^\nu}{\partial a_i^\lambda \partial a_j^\mu} = 0 \quad (\text{since } f_j^\nu \text{ does not depend on } a_i^\lambda\ \forall \lambda, i \neq j) \tag{A.10}$$

for $\mu \neq \nu$ and $\lambda \neq \nu$. For the same reason, taking $\nu = \mu$ and using equation (A.7), we have

$$\frac{\partial^2 f_i^\nu}{\partial a_i^\lambda \partial a_i^\nu} = \frac{\partial}{\partial a_i^\lambda}\left(-\frac{\partial \epsilon^i}{\partial x^i} + \frac{\partial \epsilon^j}{\partial x^j} + \frac{\partial f_j^\nu}{\partial a_j^\nu}\right) = \frac{\partial^2 f_j^\nu}{\partial a_i^\lambda \partial a_j^\nu} = 0. \tag{A.11}$$

Equations (A.10), (A.11) imply the relations

$$f_i^\nu\left(x^j, a_i^\mu\right) = C_{i\mu}^\nu(x^j) a_i^\mu + G_i^\nu(x^j) \qquad (\text{index } i \text{ not repeated}) \tag{A.12}$$

with $C_{k\mu}^\nu(x^j), G_i^\nu(x^j) \in \mathcal{F}(M)$.

Inserting equation (A.12) in equations (A.7), (A.8) we obtain that

- if $\nu \neq \mu$, then

$$C_{k\mu}^\nu = \frac{1}{m}\sum_{s=1}^m \left(C_{s\mu}^\nu\right) \Rightarrow D_\mu^\nu := C_{1\mu}^\nu = C_{2\mu}^\nu = \cdots = C_{m\mu}^\nu$$

- if $\nu = \mu$, then

$$\frac{\partial \epsilon^k}{\partial x^k} + C_{k\nu}^\nu = \frac{1}{m}\sum_{s=1}^m \left(\frac{\partial \epsilon^s}{\partial x^s} + C_{s\nu}^\nu\right) \Rightarrow D_\nu^\nu := \frac{\partial \epsilon^1}{\partial x^1} + C_{1\nu}^\nu$$
$$= \frac{\partial \epsilon^2}{\partial x^2} + C_{2\nu}^\nu = \cdots = \frac{\partial \epsilon^m}{\partial x^m} + C_{m\nu}^\nu.$$

Therefore, going back to equation (A.12) we may write

$$f_i^\nu = \sum_{\mu=1}^m D_\mu^\nu a_i^\mu - \frac{\partial \epsilon^i}{\partial x^i} a_i^\nu + G_i^\nu.$$

Inserting this last result in equation (A.6), we end up with the expressions

$$b_i^\nu = -\frac{\partial \epsilon^k}{\partial x^i} a_k^\nu + D_\mu^\nu a_i^\mu + G_i^\nu.$$

Finally, collecting all the results, we may state that the vector fields

$$X = \epsilon^i(x^j)\frac{\partial}{\partial x^i} + \left(-\frac{\partial \epsilon^k}{\partial x^q} a_k^\mu + D_\nu^\mu(x^j) a_q^\nu + G_q^\mu(x^j)\right)\frac{\partial}{\partial a_q^\mu} \tag{A.13}$$

solve equations (A.3) which is a necessary condition for a vector field X (A.1) on E to satisfy requirement (2.27).

Conversely, a straightforward check shows that all vector fields (A.13) obey the ansatz (2.27). We then conclude that they are the only vector fields whose first jet-prolongations pass to the quotient.

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